Singular Hermitian metrics on holomorphic vector bundles

Singular Hermitian metrics on holomorphic line bundles. L a holomoephic line bundle over a complex manifold X

Definition. A singular metric on L is locally giver by

$$h = e^{-2\varphi}$$

where $\varphi \in L(U, loc)$.

- The curvature form of h is locally given by
$$\Theta = i\partial\overline{\partial}\varphi$$
which is a $(1,1)$ current.
- h is called positive if Θ is positive $(1,1)$ current.
- The multiplier ideal sheaf $\mathscr{I}(h) \subset \mathscr{O}_X$ associated to a singular Hermitian metric h is defined as: $\mathscr{I}(h)_x$ consists of germs of holomorphic function $f \in \mathscr{O}_{X,x}$ such that
$$|f|^2 e^{-2\varphi}$$
is integrable in a neighuborhood of x.
- $\mathscr{D}^{p,p}(X)$
- $\alpha_1, \alpha_2 \cdots, \alpha_{n-1}$ $(1,0)$ with compact supports
- $\Theta \wedge i\alpha_1 \wedge \overline{\alpha}_1 \wedge \cdots \wedge i\alpha_{n-1} \wedge \overline{\alpha}_{n-1} = T idz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge idz_n \wedge d\overline{z}_n$, $T(f) \geq 0$

Θ is closed positive $(1,1)$ current if and only if $T = i\partial\overline{\partial}\varphi$ where φ is p.s.h locally.

If h is a smooth metric ,$\mathscr{I}(h) = \mathscr{O}_X$

Theorem 1. *If h is positive, $\mathscr{I}(h)$ is a coherent ideal sheaf.*

Proof. Use L^2 existence theorem. ☐

Theorem 2. *(Nadel vanishing theorem) (X, w) weakly pseudoconvex , L admits a singular Hermitian metric h such that*

$$\Theta_h(L) \geq \epsilon\omega$$

where ϵ is a positive continuous function on X. We have

$$H^q(X, K_X \otimes L \otimes \mathscr{I}(h)) = 0$$

for $q > 0$.

- sections of L^m iduces positive singular hermtian metric on L.

Aim: Prove similar results in the setting of vector bundles.

Cataldo, 98. Consider singular Hermtian metrics on vector bundles obtain effective results about global generated poblmes about

$$K_X^a \otimes L^b \otimes E$$

- Find a_0, b_0such that
$$K_X^a \otimes L^b \otimes E$$
are globally generated (seperated) for $a = 1, 2, b \geq b_0$.

Theorem. *(Main theorem) Let X be a projective manifold of dimension n, $E \to X$ a holomorphic vector bundle of rank r,$L \to X$ an ample line bubdle.. $N = \min\{n, r\}$. Asumme that E is $N - nef$ and p is a positive integer, If $m \geq m_1(n,p) = \frac{1}{2}(n^2 + 2pn - n + 2)$, then for any set of p distinct points $\{x_1, x_2, \cdots, x_p\}$ of X , the global sections of $K_X \otimes L^m \otimes E$ can seperate $x_1, x_2, \cdots, x_p$.*

t-nef vector bundes. V, W complex vector spaces, a tensor $\tau \in V \otimes W$ can be intepreted as linear maps

$$\alpha_\tau : V^* \to W$$

or

$$\beta_\tau : W^* \to V$$

We define the rank $\rho(\tau)$ of τas

$$\rho(\tau) = rank(\alpha_\tau) = rank(\beta_\tau)$$

- $\tau \in V \otimes W$ is decomposible, τ can be written as $\tau = v \otimes w, v \in V, w \in W$ if and only if $\rho(\tau) \leq 1$

DEFINITION 3. θ_1, θ_2 are Hermitian forms on $V \otimes W$, t is a positive number, we say

$$\theta_1 \geq_t \theta_2, (\theta_1 >_t \theta_2)$$

if $\theta_1 - \theta_2$ is semipositive (positive-definite) on all tensor $\tau \in V \otimes W$ with $\rho(\tau) \leq t$

$\rho(\tau) \leq t : \tau = \sum a_i v_i \otimes w_i, \dim L(v_i) \leq t, \dim L(w_i) \leq t$

Let E be a holomorphic vector bundle over X, h a hermitian metric on E, the curvature form

$$\Theta_h(E) \in H^0(X, \Omega^{1,1}(Hom(E,E)))$$

it can be intepreted as a Hermitian form on the bundle $T_X \otimes E$.

- L is nef if and only if for any $\epsilon > 0$，L admits a $Hermitian$metric h_ϵ such that
$$\Theta_{h_\epsilon}(L) \geq -\epsilon\omega$$

DEFINITION 4. (1) E is called t -positive (semi-positive) if E admits a Hermitian metric h such that

$$\Theta_h(E) >_t 0, (\Theta_h(E) \geq_t 0)$$

(2) E is called $t - nef$ if for any $\epsilon > 0$，E admits a Hermitian metric h_ϵ such that

$$\Theta_{h_\epsilon}(E) \geq_t -\epsilon\omega \otimes Id_{E_{h_\epsilon}}$$

where ω is a hermitian metric on X.

- $1 - positive =$ Griffiths positive
- $N - positive$= Nakano positive ,$N = \min\{n, r\}$

Singular Hermitian metric on vector bundles. From now on, we assume that X is a complex manifold of dimension n and $E \to X$ a holomorphic vector bundle of rank r.We denote by $N = \min\{n, r\}$.

DEFINITION 5. h is called a measurable metric on E if h is a section of the smooth vector bundle $E^* \otimes \overline{E}^*$with measurable corfficients and h is a almost everythere positive-definite Hermitian form on E.

- in local holomorphic frame over U, $h = (h_{ij})_{r\times r}$, h_{ij} are measurable functions on U
- h is a smooth hermitian metric, the curvature $\Theta_h(E)$ is given by

$$\Theta_h(L) = -\partial\overline{\partial}h \cdot h^{-1} + \partial h \wedge h^{-1}\overline{\partial}h \cdot h^{-1}$$

- Even if assume the measurable metric has L_{loc} coefficients.

Given a measurable metric h on E , one can associate four subsheaves of $\mathscr{O}(E)$:

- $\mathscr{O}(E) \otimes \mathscr{I}(h) \subset \mathscr{O}(E)$: here the multiplier ideal $\mathscr{I}(h) \subset \mathscr{O}_X$ is the analytic sheaf of germs of holomorphic functions on X such that $\mathscr{I}(h)_x$ consists of germs $f_x \in \mathscr{O}_{X,x}$ such that $|f_x s_x|_h^2$ is intebrable in a neighborhood of x for any $s_x \in \mathscr{O}(E)_x$. locally E has a holomorphic frames $e_1, \cdots, e_r$, $|f_x e_i|_h^2$ is integrable in a neighborhood of x for $1 \leq i \leq r$
- $\mathscr{E}(h) \subset \mathscr{O}(E)$: for any $x \in X$ and $s_x \in \mathscr{O}(E)_x$, $s_x \in \mathscr{E}(h)_x$ if and only if $|s_x|_h^2$ is integrable in a neighborhoos of x

Moreover, we h induce a measurable metric $\tilde{h}$ on the tautological line bundle $L = \mathscr{O}_{P(E)}(1) \to P(E)$, $\mathscr{I}(\tilde{h})$ ideal sheaf of $\tilde{h}$, $\pi : P(E) \to X$

- $\pi_*\mathscr{O}(L) \otimes \mathscr{I}(\tilde{h}) \subset \mathscr{O}(E)$
- In general $\mathscr{O}(E) \otimes \mathscr{I}(h) \subset \mathscr{E}(h)$

EXAMPLE 6. $E = \triangle \times \mathbb{C}^2 \to \triangle$
measurable metric $h = diag\{e^{-2\log|z|}, e^{-4\log|z|}\}$
$f_x \in \mathscr{I}(h)_x$ $|f_x|^2 e^{-2\log|z|}, |f_x|^2 e^{-4\log|z|}$ integrable around x, $\mathscr{I}(h) = (z^2)$
$\mathscr{O}(E) \otimes \mathscr{I}(h) = z^2 \cdot \mathscr{O}_X \oplus z^2 \cdot \mathscr{O}_X$
$\mathscr{E}(h) = z \cdot \mathscr{O}_X \oplus z^2 \cdot \mathscr{O}_X$

DEFINITION 7. (X, E, h, Σ, h_s) is called a singular hermitian metric on E if

- $\Sigma \subset X$ is a closed subset of measure 0;
- h_s is a sequence of smooth hermitian metrics of $E \to X$ such that

$$\lim_{s\to\infty} h_s = h$$

in $\mathscr{C}^2$ topology on $X\backslash\Sigma$

$\Theta_h(E) = \lim \Theta_{h_s}(E)$ *on* $X\backslash\Sigma$

(E, h) is Griffith positive if and only if for any local holomorphic section s of E^*, $\log|s|_{h^*}$ is plurisubharmonic

h measurable metric, (E, h) is called positive if for any local holomorphic section s of E^*, $\log|s|_{h^*}$ is plurisubharmonic

DEFINITION 8. Let ω be a hermitian metric on complex manifold X. Let (X, E, h, Σ, h_s) be a singular Hermitian metric on E, θ a real continuous $(1, 1)$ form on X and t a positive integer. We say

$$\Theta_h(E) \geq_t^{\mu} \theta \otimes Id_{E_h}$$

If there exist a sequence of continuous hermitian froms θ_s on $T_X \otimes E$ and continuous functions λ_s, λ on X such that

(1)The hermtian metrics h_s are increasing:, i.e, for any $x \in X$ and $e \in E_x$ we have

$$|e|_{h_s} \leq |e|_{h_{s+1}}$$

(2) $\theta_s \geq_t \theta \otimes Id_{E_{h_s}}$
(3) $\Theta_{h_s}(E) \geq_t \theta_s - \lambda_s \omega \otimes Id_{E_{h_s}}$
(4) $\theta_s \to \Theta_h(E), \quad , a.e. \quad on$ X
(5) $\lambda_s \to 0$ a.e. on X
(6) $0 \leq \lambda_s \leq \lambda$

- (E, h) is a Nakano positive vector bundle over a Kahler manifold (X, ω), we have
$$\Theta_h(E) \geq^{\mu}_{N} \epsilon\omega \otimes Id_{E_h}$$
for some positive continuous function ϵ
- Motivation:
- Demailly 82 : Let $h = e^{-2\varphi}$ be a singular Hermitian metric on a line bundle L and θ a continuous real $(1, 1)$ form on X such that the curvature current T satisfying
$$T \geq \theta$$
in the sense of currents, then we can obtain a collection of data (X, L, h, Σ, h_s) such that it is a singular Hermitian metric on L with
$$\Theta_h(L) \geq^{\mu}_{1} \theta \otimes Id_{L_h}$$

Theorem 9. *(L^2 existence theorem) Let (X, ω) be a weakly pseudoconvex Kahler manifold.*

E holo vector bundle of rank r

$q > 0$

h is a singular Hermitian metric on E with

$$\Theta_h(E) \geq^{\mu}_{n-q+1} \epsilon\omega \otimes Id_{E_h}$$

for a positive continuous function ϵ on X

$L^2_{p,q}(X, E, loc)$ =the space of (p, q) forms with values in E and coefficients are locally integrable on X

Then for any $g \in L^2_{n,q}(X, E, loc)$ with

- $\overline{\partial} g = 0$
- $\int_X |g|^2_h dV_\omega < \infty$
- $\int_X \frac{1}{\epsilon}|g|^2_h dV_\omega < \infty$

There exists $f \in L^2_{n,q-1}(X, E, loc)$ such that

- $\overline{\partial} f = g$
- $\int_X |f|^2_h dV_\omega \leq \frac{1}{q}\int_X \frac{1}{\epsilon}|g|^2_h dV_\omega$

Theorem 10. *Let h be a singular Hermitian metric on E such that*

$$\Theta_h(E) \geq^{\mu}_{N} \theta \otimes Id_{E_h}$$

where $N = \min\{\dim X, r(E)\}$, then $\mathscr{E}(h) \subset \mathscr{O}(E)$ is a coherent ideal sheaf .

Theorem 11. *If (X,ω) is a weakly pseudoconvex Kahler manifold, h is singular Hermitian mertric on hol vector bundle E such that*

$$\Theta_h(E) \geq^{\mu}_{N} \epsilon\omega \otimes Id_{E_h}$$

where ϵ is a positive continuous function on X and $N = \min\{\dim X, r(E)\}$,

$$H^q(X, K_X \otimes \mathscr{E}(h)) = 0$$

for $q > 0$.

Use the above Nadel type vanising theorem, we can prove the main theorem by techniques of Siu 95.

Effective results on vector bundles.

Theorem 12. *Let X be a projective manifold of dimension n, E a holomorphic vector bundle of rank r, $N = \min\{n, r\}$。 Assume that E is N-nef and A, L are ample line bundles on X. Then we have the following:*

(1) If $m \geq m_1(n,p)$, then the global sections of $K_X \otimes E \otimes (mL)$ seperate arbitrary p distnct points of X.[For p=1 we have: If $m \geq \frac{1}{2}(n^2+n+2)$, then $K_X \otimes E \otimes (mL)$ is generated by its global sections.]
(2) If $m \geq m_2(n,p;s_1,s_2,\cdots,s_p)$, then the global sections of $2K_X \otimes E \otimes (mL)$ generate simultaneous jets of order $s_1,\cdots,s_p \in \mathbb{N}$ at arbitrary p distinct points of X. [For $p = 1$ we have : If $m \geq m_2(n,1;1)$, then the global sections of $2K_X \otimes E \otimes (mL)$ sperate arbitrry pairs of points of X and generate simultaneous jets of order 1 at an arbitrary point of X]
(3) If $m \geq m_3(n,p;s_1,\cdots,s_p)$, the global sections of
$$(pn + \sum s_i + 1)K_X \otimes E \otimes (mL) \otimes A$$
generate simultaneously jets of order $s_1,s_2,\cdots,s_p$ at arbitrary p distinct points of X.
(4) If $m \geq m_4(n)$, then the global sections of $(n+2)K_X \otimes E \otimes (mL) \otimes A$ sperate arbitrry pairs of points of X and generate simultaneous jets of order 1 at an arbitrary point of X
(5) The global sections of $E \otimes (mL)$ sperate arbitrary pairs of points of X and generate simultaneous jets of order 1 at an arbitrary point of X as soon as
$$m \geq C_n(L^n)^{3(n-2)}(n+2+\frac{L^{n-1}\cdot K_X}{L^n})^{3(n-2)(\frac{n}{2}+\frac{3}{4})+\frac{1}{4}}$$
where $C_n = (2n)^{\frac{3(n-1)-1}{2}}(n^3-n^2-n-1)^{3(n-2)(\frac{n}{2}+\frac{3}{4})+\frac{1}{4}}$.

Theorem 13. （*L^2 existence theorem*）*Let (X,ω) be a weakly pseudoconvex Kahler manifold. L is a holo line bundle over X equipped with a positive singular Hermitian metric $h = e^{-\varphi}$ such that*

$$\Theta_h(L) = i\partial\overline{\partial}\varphi \geq \epsilon\omega$$

for some $\epsilon > 0$. Denote by
$L^2_{p,q}(X, L.loc)$ = the space of (p,q) forms with values in L and coefficients which are locally integrable.
For any $g \in L^2_{n,q}(X, L, loc)$ with

$q > 0$
$\overline{\partial} g = 0$
$\int_X |g|_h^2 dV_\omega < \infty$
There exists $f \in L^2_{n,q-1}(X, L, loc)$ *such that*
$\overline{\partial} f = g$
$\int_X |f|_h^2 dV_\omega \leq \frac{1}{q\epsilon} \int_X |g|_h^2 dV_\omega$

Demailly $P39$

- A complex manifold is called weakly pseudoconvex if there exists a smooth psh ψ on X such that
$$X_c = \{x \in X : \psi(x) < c\}$$
is relatively compact in X for all $c > 0$
- compact complex manifolds are weakly pseudoconvex.

LEMMA 14. *(Grauert-Remmert , Coherent Analytic sheaf P111) Let* $\mathscr{I}$ *be a coherent analytic sheaf on a complex space* X*,*

$$\{s_i : i \in I\}$$

be a subset of sections of $\mathscr{I}$ *over* X*. Then the subsheaf* $\mathscr{I}_I$ *of* $\mathscr{I}$ *generated by* $s_i (i \in I)$ *is a coherent analytic sheaf.*

$f_x \in \mathscr{I}(h)_x$ if and only if $|f|^2 e^{-\varphi}$ is integrable in a neighborhood of x
$\varphi + C|z|^2$ φ

LEMMA 15. *(Nakayama Lemma)(A-M P22 coro 2.7)*

Let M *be a finitely generated* $A-$ *module,* N *a submodulemof* M*,* $a \subset$ *the Jacobson radical of* A*. If*

$$M = aM + N$$

we have $M = N$

LEMMA 16. *)*

(Atiyah-MacDonald coro 10.10) Let A *be a Notherian ring ,an ideal of* A*,* M *a finite generated* A *module and* $M' \subset M$ *a submodule. Then there exists an integer* k *such that*

$$(a^n M) \cap M' = a^{n-k} \cdot ((a^k M) \cap M')$$

for all $n \geq k$*.*

$A = M = \mathscr{O}_x, M' = \mathscr{I}(h)_x$

THEOREM 17. *Let* X *be a complex manifold and* L *a hol line bundle on* X*.* $h = e^{-\varphi}$ *is a singular hermitian metric on* L *such that*

$$\Theta_h(L) = i\partial\overline{\partial}\varphi \geq \theta$$

for some real continuous $(1,1)$ *form, the multiplier ideal sheaf* $\mathscr{I}(h)$ *is a coherent analytic sheaf.*

PROOF. Since coherence is a local property, we can assume
X is a ball in $\mathbb{C}^n$ centered at x
the coefficients of θ is bounded
L is trivial over X □

Let ω be the standard Kahler form on $\mathbb{C}^n$.
$S =$ the space of holomorphic functions f on X such that

$$\int_X |f|^2 e^{-\varphi} dV_\omega < \infty$$

$\mathscr{F} \subset \mathscr{O}_X$ the subsheaf of $\mathscr{O}_X$ generated by S
Then we have
$\mathscr{F}$ is a coherent analytic sheaf.(Lemma 14)
$\mathscr{F} \subset \mathscr{I}(h)$
We will prove that $\mathscr{I}(h) = \mathscr{F}$, or $\mathscr{I}(h)_x = \mathscr{F}_x$
*Step*1 For any $x \in X$ we have

$$\mathscr{F}_x + \mathscr{I}(h)_x \cap m_x^l = \mathscr{I}(h)_x$$

for every positive integer l.
For any $x \in X$, $f_x \in \mathscr{I}(h)_x$, we can choose a cut-off functio σ such that
$\sigma = 1$ in a neighborhood of x
σf_x is a smooth function on X
$\int_X |\sigma f_x|^2 e^{-\varphi} dV_\omega < \infty$
We can choose a constant $C > 0$ such that

$$\theta + C \cdot i\partial\overline{\partial}|z|^2 \geq \omega = i\partial\overline{\partial}|z|^2$$

on X.

$$i\partial\overline{\partial}(\varphi(z) + C|z|^2) \geq \omega > 0$$

Take a strictly psh function

$$\varphi_l(z) = \varphi(z) + 2(n+l)\log|z-x| + C|z|^2$$

We twist the metric h by $h_l = e^{-\varphi_l(z)}$, then we have

$$\Theta_{h_l}(L) = i\partial\overline{\partial}\varphi_l = i\partial\overline{\partial}\varphi + 2(n+l)i\partial\overline{\partial}log|z-x| + C \cdot i\partial\overline{\partial}|z|^2 \geq \omega$$

Takea $(0,1)$ form $g = \overline{\partial}(\sigma f_x)$ we have

$$\int_X |g|^2 e^{-\varphi_l} dV_\omega = \int_X |g|^2 e^{-\varphi(z)} e^{-2(n+l)\log|z-x|} e^{-C|z|^2} dV_\omega < \infty$$

By Theorem 13 there exists a locally square integrable function f on X such that

$$\overline{\partial} f = g = \overline{\partial}(\sigma \cdot f_x)$$

$$\int_X |f|^2 e^{-\varphi_l} dV_\omega \leq \int_X |g|^2 e^{-\varphi_l} dV_\omega < \infty$$

Take $F = \sigma f_x - f$, we have $\overline{\partial} F = 0$, hence F is a holomorphic function on X.

$$\sigma f_x = F + f$$

Recall that $C|z|^2$ is bounded on X, $e^{-2(n+l)log|z-x|^2} = \frac{1}{|z-x|^{2(n+l)}} \geq C_0 > 0$ on X

$$\int_X |f|^2 e^{-\varphi(z)} \leq \frac{1}{C_1} \int_X |f|^2 e^{-\varphi(z)} e^{-2(n+l)log|z-x|} e^{-C|z|^2} dV_\omega < \infty$$

Hence we have

$$\int_X |F|^2 e^{-\varphi(z)} dV_\omega < \infty$$

$f = \sigma f_x - F = f_x - F$ is holomorphic in a small neighborhood of $x$and $f \in \mathscr{I}(h)_x$

We have $\varphi(z) + C|z|^2$ is psh in a neighborhood of $\overline{X}$, Hence we have

$$\varphi(z) + C|z|^2 \leq C_3$$

on X and

$\infty > \int_X |f|^2 e^{-\varphi_l dV_\omega} \geq e^{-C_3} \int_X |f|^2 e^{-2(n+l)log|z-x|} dV_\omega$

Hence we have $f \in m_x^l$ and $f \in m_x^l \cap \mathscr{I}(h)_x$.

$$f_x = \sigma f_x = F + f \in \mathscr{F}_x + m_x^l \cap \mathscr{I}(h)_x$$

*Step*2 There exists $\gamma \geq 1$ such that

$$\mathscr{F}_x + m_x^\gamma \cdot \mathscr{I}(h)_x = \mathscr{I}(h)_x$$

*By Lemma*15 we have

$\mathscr{I}(h)_x = \mathscr{F}_x + \mathscr{I}(h)_x \cap m_x^l$

$\subset \mathscr{F}_x + m_x^{l-k} \cdot \mathscr{I}(h)_x$

$\subset \mathscr{F}_x + m_x \cdot \mathscr{I}(h)_x$

$= \mathscr{I}(h)_x$

It follows that

$$\mathscr{F}_x + m_x^\gamma \cdot \mathscr{I}(h)_x = \mathscr{I}(h)_x$$

*By Lemma*15 we have

$$\mathscr{I}(h)_x = \mathscr{F}_x$$

THEOREM 18. *(Nadel vanishing theorem)*

Let (X, ω) be a weakly pseudoconvex Kahler manifold. L is a holo line bundle over X equipped with a positive singular Hermitian metric $h = e^{-\varphi}$ such that

$$\Theta_h(L) = i\partial\overline{\partial}\varphi \geq \epsilon\omega$$

for some $\epsilon > 0$. Then

$$H^q(X, K_X \otimes L \otimes \mathscr{I}(h)) = 0$$

for $q > 0$.

PROOF. We choose a resolution of $K_X \otimes L \otimes \mathscr{I}(h)$ in the following way:

$\mathscr{L}^q$ =sheaf of germs of (n, q) forms u with values in L with mearable coefficients such that

$$|u|_h^2, |\overline{\partial}u|_h^2$$

are locally integrable.

$\overline{\partial}$ can act on $\mathscr{L}^q$, $\overline{\partial}:\mathscr{L}^q\to\mathscr{L}^{q+1}$;

$\mathscr{L}^q$ are modules over the structure sheaf of smooth manifold X, hence $H^k(X,\mathscr{L}^q)=0$ for $k>0$.

$Ker\{\overline{\partial}:\mathscr{L}^0\to\mathscr{L}^1\}=K_X\otimes L\otimes\mathscr{I}(h)$

Apply L^2 existence theorem to a ball centered at $x\in X$, we have that the sequence

$$0\to K_X\otimes L\otimes\mathscr{I}(h)\to\mathscr{L}^0\to\mathscr{L}^1\cdots\mathscr{L}^n\to 0$$

is exact and $H^k(X,\mathscr{L}^q)=0$ for $k\geq 1$. Hence we have

$$H^q(X,K_X\otimes L\otimes\mathscr{I}(h))=\frac{Ker\{\overline{\partial}:\mathscr{L}^q(X)\to\mathscr{L}^{q+1}(X)\}}{Im\{\overline{\partial}:\mathscr{L}^{q-1}(X)\to\mathscr{L}^q(X)\}}$$

Let $\psi\in PSH(X)$ be a smooth exhaustion function on X.

For any $g\in\mathscr{L}^q(X)$ with $\overline{\partial}g=0$, we can choose a smooth convex function $\chi:\mathbb{R}\to\mathbb{R}$ ($\lim_{t\to\infty}\chi(t)=0$fast enough) such that

$$\int_X |g|_h^2 e^{-\chi(\psi)}dV_\omega<\infty$$

Take $\tilde{h}=he^{-\chi(\psi)}$ we have

$$\Theta_{\tilde{h}}(L)=\Theta_h(L)+i\partial\overline{\partial}\chi(\psi)\geq\epsilon\omega$$

By L^2 existence theorem there exists $f\in L^2_{(n,q-1)}(X,L,loc)$ such that

$\overline{\partial}f=g$

$\int_X |f|_h^2 e^{-\chi(\psi)}dV_\omega<\infty$

For $x\in X$, $e^{-\chi(\psi)}$ is lower bounded by positive number, hence $|f|_h^2$ is intebrale in a neighborhood of x, hence

$$f\in\mathscr{L}^{q-1}(X)$$

and

$$H^q(X,K_X\otimes L\otimes\mathscr{I}(h))=0$$

□